involving the R and Q vanish at times longer than a transit time) and so these quantities are given only to within constant factors (although the factors for $f_{e;1}(z)$ and $f_{h;1}(z)$ are related). The values of these factors are determined by the initial conditions contained in the source terms, which do not appear in this asymptotic analysis. However, Eqs. (8) and (9) are nonlinear and the equations for the higher order coefficients, $f_{e,h;n}(z)$, with $n = 2, 3, \ldots$, involve the lower order coefficients so that these can be determined successively after the $f_{e,h;1}(z)$ are known.

We now determine the coefficients $f_{e,h;1}(z)$. For simplicity, to derive expressions for the coefficients, $f_{e,h;1}(z)$ of the leading terms of the Laurent expansion in Eq. (10) for $f_{e,h}(z)$, we use the local, constant velocity model. The ionization event pdfs then become

$$h_e(\zeta,\tau) = \alpha \exp(-\alpha\zeta)\delta(\tau - \zeta/v_e) \text{ and } h_h(\zeta,\tau) = \beta \exp(-\beta\zeta)\delta(\tau - \zeta/v_h). \quad (11)$$

Using this model it is convenient to write Eqs. (8) and (9) in terms of dimensionless length variables, $s = z/w$ and $p = \zeta/w$ and dimensionless time variable, $r = t/\tau_0$, where $\tau_0 = (\tau_e + \tau_h)/2$ and $\tau_{e,h} = w/v_{e,h}$ are the electron and hole transit times across the multiplication region. Writing $\varphi(s,r) \equiv f_{e,h}(z,t)$, Eqs. (8) and (9) become

$$\begin{aligned} \varphi_e(s,r) &= \exp(a(s-1))\theta(1-s-r/\rho_e) \\ &+ a\int_0^{1-s} \exp(-ap)\big(2\varphi_e + \varphi_h - \varphi_e^2 - 2\varphi_e\varphi_h + \varphi_e^2\varphi_h\big)dp \end{aligned} \quad (12)$$

and

$$\begin{aligned} \varphi_h(s,r) &= \exp(b(s-1))\theta(1-s-r/\rho_b) + b\int_0^{1-s} \exp(-bp)\big(2\varphi_h \\ &+ \varphi_e - \varphi_h^2 - 2\varphi_h\varphi_e + \varphi_h^2\varphi_e\big)dp\,. \end{aligned} \quad (13)$$

Here $\theta(x)$ is the unit step function, $\rho_{e,h} = \tau_{e,h}/\tau_0$, the $\varphi_{e,h}$ on the right-hand side (RHS) of Eq. (12) are all to be understood as $\varphi_{e,h}(s+p, r-\rho_e p)$ and on the RHS of Eq. (13) as $\varphi_{e,h}(s-p, r-\rho_h p)$. The solutions $\varphi_{e,h}$ are evidently determined by the values of the dimensionless parameters, $a = \alpha w$, $b = \beta w$ and $\rho_{e,h}$, which are all of order unity (indeed, when $\alpha = \beta$ and $v_e = v_h$ then they are precisely unity, since we are considering a device biased at breakdown). The $\varphi_{e,h}(s,r)$ themselves are dimensionless since they represent probabilities.

Equations for the coefficients $f_{e,h;1}(z) \equiv \varphi_{e,h}(s)$ can now be found by discarding all terms nonlinear in the $\varphi_{e,h}$ on the RHS of Eq. (12) (since they do not contribute to the leading terms in the Laurent expansion) and writing $\varphi_{e,h}(s,r) \sim \varphi_{e,h}(s)/r$ on the left and $\varphi_{e,h}(s \pm p, r \pm \rho_{e,h}p) \sim \varphi_{e,h}(s \pm p)/r$ on the right, since at long times $r \gg \rho_{e,h}p$. The inhomogeneous terms also disappear at long times and we find

$$\varphi_e(s) = a\int_0^{1-s} \exp(-ap)\big(2\varphi_e(s+p) + \varphi_h(s+p)\big)dp \quad (14)$$

and

$$\varphi_h(s) = b\int_0^{s} \exp(-bp)\big(2\varphi_h(s-p) + \varphi_h(s-p)\big)dp\,. \quad (15)$$

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Changing the integration variable in Eq. (14) to $u = s + p$ and in Eq. (15) to $u = s - p$ these equations become

$$\exp(-as)\varphi_e(s) = a\int_0^1 \exp(-au)\big(2\varphi_e(u) + \varphi_h(u)\big)du \tag{16}$$

and

$$\exp(bs)\varphi_h(s) = b\int_0^s \exp(bu)\big(2\varphi_h(u) + \varphi_e(u)\big)du\,. \tag{17}$$

By differentiating these equations with respect to s we can find differential equations for the $\varphi_{e,h}(s)$. With the boundary conditions, $\varphi_e(1) = 0 = \varphi_h(0)$ these yield solutions which we write as

$$\varphi_e(s) = \frac{C}{d}\big(\exp(d(1-s)) - 1\big) \text{ and } \varphi_h(s) = \frac{C}{d}\big(1 - \exp(-ds)\big). \tag{18}$$

Here $d = a - b$ and C is a dimensionless constant, determined by the values of a, b and via Eq. (12). Numerical solutions of Eqs. (6) and (7) confirm the $1/t$ behavior of the $f_{e,h}(z,t)$ and show that C is of the order of unity. We are indebted to C. H. Tan for these results. The d in the denominator of Eq. (18) is included to preserve good behavior as $d \to 0$. The analysis also yields the breakdown condition, $b\exp(a) = a\exp(b)$, confirming that these arguments are valid only at breakdown threshold and not above or below.

Finally, writing $\delta = \alpha - \beta$ we can deduce the form of the leading terms of Laurent expansions. Thus, since $f_{e;1}(z)/t = j_e(s)/r$, and $f_{h;1}(z)/t = j_h(s)/r$ it follows that

$$f_{e;1}(z) = \frac{C\tau_0}{\delta w}\big(\exp(\delta(w-z)) - 1\big) \text{ and } f_{h;1}(z) = \frac{C\tau_0}{\delta w}\big(1 - \exp(-\delta z)\big). \tag{19}$$

The dynamical equations for the avalanche carrier densities in a uniform multiplication region are given, e.g., by Emmons [26]. In terms of the electron and hole concentrations per unit length, $n(z)$ and $p(z)$. They are given by

$$n(z) = \frac{I\big(1 - \exp(\delta z)\big)}{qv_e\big(\exp(\delta z) - 1\big)} \text{ and } p(z) = \frac{I\big(\exp(\delta z) - \exp(\delta w)\big)}{qv_h\big(\exp(\delta w) - 1\big)}, \tag{20}$$

where I is the current carried by these distributions.

We now determine the probability distribution function of the quenching time in a passively quenched SPAD under constant electric field at breakdown. To calculate the statistics of the duration of such avalanche pulses we observe that the mean avalanche current is generated by electrons and holes distributed throughout the multiplication region. In the local model we can regard these as primary carriers, each generating its own individual avalanche current, all of which flow in parallel to generate a total mean current, I. For the avalanche pulse to quench each of these individual avalanche currents must terminate independently. The probability that this happens before time t elapses is given by $F_I(t) = \prod_i F_{e,h}(z_i,t)$, where the product is over all electrons and holes, situated at z_i in the multiplication region, and $F_{e,h}(z_i,t)$ is the probability that

#156323 - $15.00 USD Received 11 Oct 2011; revised 6 Dec 2011; accepted 19 Dec 2011; published 10 Jan 2012

an electron (hole) injected at z_i will give rise to an avalanche pulse which terminates before time t has elapsed. When the SPAD is biased precisely at breakdown (i.e., prior to the avalanche current collapsing) $F_{e,h}(z_i,t) = 1 - f_{e,h}(z_i)/t$. Interestingly, we note that this asymptotic behavior is different from those corresponding to below or above breakdown, for which the asymptotic behavior is exponential [11, 20]. Thus, at times long compared with the carrier transit times we find

$$\ln\bigl(F_I(t)\bigr) = \sum_i \ln\bigl(1 - f_{e,h}(z_i)/t\bigr) \approx -\frac{1}{t}\sum_i f_{e,h}(z_i). \tag{21}$$

If the electron and hole distributions per unit length in the multiplication region are $n(z)$ and $p(z)$ then Eq. (21) becomes

$$\ln\bigl(F_I(t)\bigr) \approx -\frac{1}{t}\left(\int_0^w n(z) f_{e;1}(z)dz + \int_0^w f_{h;1} p(z)dz\right). \tag{22}$$

By using the expressions for the $f_{e,h;1}(z)$ given by Eq. (19), and the expressions for the $n(z)$ and $p(z)$ given by Eq. (20), respectively, we arrive at Eq. (1).

Acknowledgments

We thank CH Tan for numerical solutions of Eqs. (6) and (7). This work was supported in part by NASA under Grant NNG06LA04C.